

The Coulomb Green's Function in Two Dimensions

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Abstract

We consider the two-dimensional non-relativistic Coulomb problem with the aid of the momentum space construction of the associated Green's function. Our presentation has precursors in three dimensions. It is mainly Schwinger's approach which we advocate and project onto two dimensions.

1 Introduction

The quantum mechanical Coulomb problem plays a central role in atomic physics. Its solution is commonly studied using Schrödinger's wave equation, wherein the bound state spectrum as well as the scattering states can be exhibited by employing either spherical or parabolic coordinates. One should not forget, however, that it was Pauli who gave the first solution of the hydrogen atom with the aid of the Laplace-Runge-Lenz vector which makes the H-atom a dynamical symmetry problem in group theory. Finally there is the powerful method of Green's functions which contains all information about the system. For our case it means that if one is able to present a closed-form expression for the Coulomb Green's function, one can immediately extract the energy spectrum as well as the wave functions. Hence we have to solve the Green's function equation or the associated integral equation for the Coulomb potential.

In the sequel it is our goal to help the reader to analyze the Coulomb problem once again using the language of Green's functions, but this time in two spatial dimensions. Admittedly the three dimensional problem is the physically important one and technically somewhat more complicated than the two-dimensional case; it seems to us, however, that a pedagogical discussion of the two-dimensional Green's function for the Coulomb problem should be of wide interest.

Our treatment will be reminiscent of Schwinger's paper¹. There are, however, calculations based on the hydrogen wave function in momentum space that date back to V.A. Fock² and V. Bargmann³. The paper by B. Podolsky and L. Pauling⁴ is yet another important contribution. Along the same line of thought are the articles furnished by the authors

in Ref. 5 and 6. Two more recent pedagogically noteworthy articles were provided by B.R. Holstein⁷ and G.S. Adkins⁸. The nice review article by M. Lieber⁹ and his contribution in Ref. 10 is also worth mentioning. In the beginning there stands, of course, Pauli's seminal work on the hydrogen atom¹¹.

2 The 2-D Hydrogen Atom in Momentum Space and its Projection onto the Fock Sphere in 3-D

Since the problem has already been discussed in this Journal¹² we will merely list some of the well-known results following from the existence of the conserved Laplace-Runge-Lenz vector. But we will also remind the reader of Fock's and Bargmann's work in the context of the simpler two-dimensional Coulomb problem and so assist the student in understanding their contribution as well.

In two dimensions, the Hamiltonian is given by

$$H = \frac{\mathbf{p}^2}{2m} - \frac{Ze^2}{r}, \quad \mathbf{p}^2 = p_x^2 + p_y^2, \quad r = \sqrt{x^2 + y^2}. \quad (1)$$

The angular momentum vector has only one component, $L = L_3$, and the Runge-Lenz vector degenerates to a two-dimensional vector, $\mathbf{A} = (A_1, A_2)$. In 2-D one finds ($\hbar = 1$):

$$\mathbf{L} \times \mathbf{p} + \mathbf{p} \times \mathbf{L} = i\mathbf{p} \quad (\text{not } 2i\mathbf{p} \text{ as in 3-D}), \quad (2)$$

so that

$$\mathbf{A} = \frac{\mathbf{r}}{r} + \frac{1}{mZe^2} \left(-\mathbf{p} \times \mathbf{L} + \frac{1}{2}i\mathbf{p} \right) \quad (3)$$

and

$$\mathbf{A} \times \mathbf{A} = i \frac{(-2H)}{mZ^2e^4} \mathbf{L}, \quad (4)$$

i.e.,

$$[A_1, A_2] = i \frac{(-2H)}{mZ^2e^4} L. \quad (5)$$

We are interested in the bound state spectrum ($-H > 0$) with the energy values

$$E' = -\frac{mZ^2e^4}{2\left(l + \frac{1}{2}\right)^2}, \quad l = 0, 1, 2, \dots, \quad l + \frac{1}{2} =: \nu, \quad (6)$$

so that we obtain for energy eigenstates according to Eq. (4)

$$\nu \mathbf{A} \times \nu \mathbf{A} = i \mathbf{L}. \quad (7)$$

Recall that in 2-D the vector product is a pure number:

$$\mathbf{a} \times \mathbf{b} = a_1 b_2 - a_2 b_1 = \#.$$

Now it is useful to eliminate $\frac{1}{r}$ in favor of \mathbf{p}^2 and H :

$$\frac{1}{r} = \frac{1}{Ze^2} \left(\frac{\mathbf{p}^2}{2m} - H \right) = \frac{1}{mZe^2} \left(\frac{\mathbf{p}^2}{2} - mH \right). \quad (8)$$

Then Eq. (3) can be rewritten as

$$\begin{aligned} mZe^2 \mathbf{A} &= \mathbf{r} \frac{mZe^2}{r} - \left(\underbrace{\mathbf{p} \times \mathbf{L}} - \frac{1}{2} i \mathbf{p} \right) \\ &= -\mathbf{p} \times (\mathbf{p} \times \mathbf{r}) = -\mathbf{p} \mathbf{p} \cdot \mathbf{r} + \mathbf{p}^2 \mathbf{r} \\ &= -\mathbf{r} mH + \frac{1}{2} \mathbf{r} \mathbf{p}^2 + \mathbf{p} \mathbf{p} \cdot \mathbf{r} - \mathbf{p}^2 \mathbf{r} + \frac{1}{2} i \mathbf{p}. \end{aligned} \quad (9)$$

When acting on energy eigenstates one may write

$$H = E = -\frac{mZ^2e^4}{2\nu^2} = -\frac{\left(\frac{Z}{a_0}\right)^2}{2m} \frac{1}{\nu^2}, \quad a_0 = \frac{\hbar^2}{me^2}. \quad (10)$$

Introducing the effective momentum

$$p_0 = \frac{Z}{a_0} \frac{1}{\nu}, \quad (11)$$

we can replace Eq. (10) by

$$-E = \frac{p_0^2}{2m}, \quad (12)$$

which yields for mH in Eq. (9): $-mH = \frac{p_0^2}{2}$. We also rewrite in Eq. (9)

$$\frac{1}{2} \mathbf{r} \mathbf{p}^2 = \frac{1}{2} \mathbf{p}^2 \mathbf{r} + \frac{1}{2} [\mathbf{r}, \mathbf{p}^2] = \frac{1}{2} \mathbf{p}^2 \mathbf{r} + i \mathbf{p},$$

so that we obtain

$$mZe^2 \mathbf{A} = \frac{1}{2}(p_0^2 - p^2)\mathbf{r} + \mathbf{p} \mathbf{p} \cdot \mathbf{r} + \frac{3}{2}i\mathbf{p}. \quad (13)$$

In 3-D one finds 2 instead of $\frac{3}{2}$ in the last term on the right-hand side of Eq. (13).

At this stage we go to the momentum representation and write for any state vector $|\rangle$ the numerical value

$$\langle \mathbf{p} | \mathbf{A}(\mathbf{r}, \mathbf{p}) | \rangle = \mathbf{A} \left(i \frac{\partial}{\partial \mathbf{p}}, \mathbf{p} \right) \langle \mathbf{p} | \rangle \equiv \mathbf{A} \psi(\mathbf{p}). \quad (14)$$

Consequently, the operator statement (13) turns into a differential equation:

$$mZe^2 \mathbf{A} \psi = \left[\frac{1}{2}(p_0^2 - p^2) i \frac{\partial}{\partial \mathbf{p}} + \mathbf{p} \mathbf{p} \cdot i \frac{\partial}{\partial \mathbf{p}} + \frac{3}{2}i\mathbf{p} \right] 1 \psi. \quad (15)$$

We now put $1 = \frac{1}{(p_0^2 + p^2)^{3/2}}(p_0^2 + p^2)^{3/2}$ and pull the denominator all the way to the left, past the differential operator. Here we employ the formula

$$\frac{\partial}{\partial \lambda} f = f \frac{\partial}{\partial \lambda} + \left(\frac{\partial f}{\partial \lambda} \right) = f \frac{\partial}{\partial \lambda} + f \frac{\partial f}{f \partial \lambda} = f \left(\frac{\partial}{\partial \lambda} + \frac{\partial}{\partial \lambda} \log f \right),$$

and upon using $f = \frac{1}{(p_0^2 + p^2)^{3/2}}$ so that $\frac{\partial}{\partial \mathbf{p}} \log \frac{1}{(p_0^2 + p^2)^{3/2}} = -3 \frac{\mathbf{p}}{(p_0^2 + p^2)}$, we obtain for Eq. (15):

$$p_0 \nu \mathbf{A} \psi = \frac{1}{(p_0^2 + p^2)^{3/2}} i \left[\frac{1}{2}(p_0^2 - p^2) \frac{\partial}{\partial \mathbf{p}} + \mathbf{p} \mathbf{p} \cdot \frac{\partial}{\partial \mathbf{p}} \right] (p_0^2 + p^2)^{3/2} \psi. \quad (16)$$

On the left-hand side of this equation we made use of (11): $mZe^2 = \frac{Z}{a_0} = \nu p_0$.

The result (16) can be checked for the ground state $\mathbf{A} = 0$, which requires $(p_0^2 + p^2)^{3/2} \psi_0 = \text{const.}$ or $\psi_0(\mathbf{p}) = \frac{\text{const.}}{(p_0^2 + p^2)^{3/2}}$. This is also the wave function found in Ref. 13, where the two-dimensional analog of Fock's treatment is exhibited beautifully.

In the sequel we will be interested in commutation relations $[X, Y] = Z$. Say we have $\bar{X} = F^{-1} X F$, which is called a similarity transformation, maintaining algebraic properties. We encounter such a situation in Eq. (16) with $F = (p_0^2 + p^2)^{3/2}$. If we want simple commutation relations, we must look in the middle, i.e., in the square brackets of (16), as the wings cannot be effective. To do so we consider a change of variables to eliminate the $\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{p}}$ term in (16), which can be accomplished by projecting the 2-D momentum space onto the 3-D sphere in the same way that Fock did for the 3-D momentum space problem. Hence we introduce the stereographic projection.

$$\boldsymbol{\xi} = \frac{2p_0 \mathbf{p}}{\lambda(p)}, \quad \xi_0 = \frac{p_0^2 - p^2}{\lambda(p)}, \quad \lambda(p) := p_0^2 + p^2, \quad (17)$$

where $\xi = (\xi_0, \boldsymbol{\xi})$ is a unit 3-vector,

$$\boldsymbol{\xi}^2 + \xi_0^2 = 1, \quad (18)$$

i.e., defines the unit sphere in a 3-D Euclidean space. In Eq. (16) we need

$$\frac{\partial}{\partial \mathbf{p}} = \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{p}} \frac{\partial}{\partial \boldsymbol{\xi}} = 2p_0 \left(\frac{1}{\lambda(p)} - \frac{2\mathbf{p} \cdot \mathbf{p}}{\lambda(p)^2} \right) \frac{\partial}{\partial \boldsymbol{\xi}}.$$

Then $\mathbf{p} \cdot \mathbf{p} \frac{\partial}{\partial \mathbf{p}} = 2p_0 \mathbf{p} \frac{p_0^2 - p^2}{\lambda(p)^2} \mathbf{p} \cdot \frac{\partial}{\partial \boldsymbol{\xi}}$.

Using these results in (16) we obtain

$$\frac{1}{2}(p_0^2 - p^2) \frac{\partial}{\partial \mathbf{p}} + \mathbf{p} \cdot \mathbf{p} \frac{\partial}{\partial \mathbf{p}} = p_0 \frac{p_0^2 - p^2}{\lambda(p)} \frac{\partial}{\partial \boldsymbol{\xi}} = p_0 \xi_0 \frac{\partial}{\partial \boldsymbol{\xi}}. \quad (19)$$

Now Eq. (16) can be expressed in the form

$$\nu \mathbf{A} \psi = \frac{1}{\lambda(p)^{3/2}} i \xi_0 \frac{\partial}{\partial \boldsymbol{\xi}} \lambda(p)^{3/2} \psi, \quad (20)$$

$$\text{i.e.,} \quad \nu \mathbf{A} \rightarrow i \xi_0 \frac{\partial}{\partial \boldsymbol{\xi}}, \quad (21)$$

Because $\lambda(p)^{3/2}$ is a scalar we can also write \mathbf{L} in terms of $\boldsymbol{\xi}$:

$$(\mathbf{L})_3 \rightarrow \frac{1}{i} \left(\mathbf{p} \times \frac{\partial}{\partial \mathbf{p}} \right)_3 = \frac{1}{i} \left(\boldsymbol{\xi} \times \frac{\partial}{\partial \boldsymbol{\xi}} \right)_3 = \lambda(p)^{-3/2} \frac{1}{i} \underbrace{\left(\boldsymbol{\xi} \times \frac{\partial}{\partial \boldsymbol{\xi}} \right)_3}_{= \xi_1 \partial_2 - \xi_2 \partial_1} \lambda(p)^{3/2}. \quad (22)$$

So, except for a similarity transformation in (20), we have

$$\begin{aligned} [\nu A_1, \nu A_2] &= - \left[\xi_0 \frac{\partial}{\partial \xi_1}, \xi_0 \frac{\partial}{\partial \xi_2} \right] \\ &= -\xi_0 \left[\frac{\partial}{\partial \xi_1}, \xi_0 \right] \frac{\partial}{\partial \xi_2} - \xi_0 \left[\xi_0, \frac{\partial}{\partial \xi_2} \right] \frac{\partial}{\partial \xi_1}. \end{aligned} \quad (23)$$

From the constraint equation $\boldsymbol{\xi}^2 + \xi_0^2 = 1$ we can use $\frac{\partial}{\partial \boldsymbol{\xi}} \xi_0 = -\frac{\boldsymbol{\xi}}{\xi_0}$. Accordingly Eq. (23) reduces to

$$[\nu A_1, \nu A_2] = -\xi_0 \left(-\frac{\xi_1}{\xi_0} \right) \frac{\partial}{\partial \xi_2} - \xi_0 \left(\frac{\xi_2}{\xi_0} \right) \frac{\partial}{\partial \xi_1} = \xi_1 \partial_2 - \xi_2 \partial_1 =: i L_{12} \equiv i L_3. \quad (24)$$

\mathbf{A} in (21) and \mathbf{L} on the right-hand side of (24) look quite different. But it is possible to write them in the same form. To do this we must get off the Fock sphere, which is our unit sphere in 3-D space. Recall that, up until now, ξ_0 has not been an independent variable: $\xi_0^2 = 1 - \boldsymbol{\xi}^2$. Now let us think of ξ_0 as being independent. Then the following obvious relation exists between our former spatial derivative, where ξ_0 was constrained, and a new derivative, where ξ_0 is now an independent variable:

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{\xi}}(\xi_0 \text{ constrained}) &\rightarrow \frac{\partial}{\partial \boldsymbol{\xi}} + \frac{\partial \xi_0}{\partial \boldsymbol{\xi}} \frac{\partial}{\partial \xi_0}(\xi_0 \text{ independent}) \\ &= \frac{\partial}{\partial \boldsymbol{\xi}} - \frac{\boldsymbol{\xi}}{\xi_0} \frac{\partial}{\partial \xi_0}. \end{aligned}$$

Now we can write, instead of Eq. (21), where ξ_0 is still a dependent variable,

$$-\xi_0 \frac{1}{i} \frac{\partial}{\partial \boldsymbol{\xi}}(\xi_0 \text{ dep. variable}) \rightarrow \boldsymbol{\xi} \frac{1}{i} \frac{\partial}{\partial \xi_0} - \xi_0 \frac{1}{i} \frac{\partial}{\partial \boldsymbol{\xi}}(\xi_0 \text{ indep. variable}). \quad (25)$$

Eq. (25) is just a rotation connecting the 0-axis with the k -th axis ($k = 1, 2$). This, then, is the meaning of \mathbf{A} as a generator of rotation. Our whole algebra becomes evident when we write

$$L_3 = L_{12}, \quad \nu A_1 =: L_{20}, \quad \nu A_2 =: L_{01}, \quad (26)$$

$$\text{then} \quad L_{ab} = \xi_a \frac{1}{i} \frac{\partial}{\partial \xi_b} - \xi_b \frac{1}{i} \frac{\partial}{\partial \xi_a}, \quad a, b = 0, 1, 2, \quad (27)$$

and a direct calculation yields

$$\frac{1}{i} [L_{ab}, L_{cd}] = \delta_{ad} L_{cb} - \delta_{bd} L_{ca}. \quad (28)$$

So we have found

$$\nu \mathbf{A} \psi(p) = \frac{1}{(p_0^2 + p^2)^{3/2}} \mathbf{M} (p_0^2 + p^2)^{3/2} \psi(p), \quad (29)$$

where \mathbf{M} is the differential operator

$$\mathbf{M} = \boldsymbol{\xi} \frac{1}{i} \frac{\partial}{\partial \xi_0} - \xi_0 \frac{1}{i} \frac{\partial}{\partial \boldsymbol{\xi}}, \quad (30)$$

where $\boldsymbol{\xi}$ and ξ_0 are given by Eq. (17). If we then put

$$M_1 = \nu A_1 = L_{20}, \quad M_2 = \nu A_2 = L_{01}, \quad L_3 = L_{12}, \quad (31)$$

or $L := (\mathbf{M}, L_3)$, we obtain the O(3) algebra

$$\frac{1}{i}(L \times L) = L, \quad (32)$$

$$L^2 = L_{12}^2 + L_{20}^2 + L_{01}^2 \equiv L_3 + \mathbf{M}^2 = \sum_{a,b} L_{ab}^2. \quad (33)$$

We know the eigenvalues of L^2 with L satisfying (32):

$$(L^2)' = (\hbar^2) l(l+1), \quad l = 0, 1, 2, \dots \quad (34)$$

The eigenfunctions are, of course, the spherical harmonics $Y_{lm}(\Omega)$. So we obtain

$$L^2 Y_{lm} = l(l+1) Y_{lm}, \quad (35)$$

and the quantum number m can take all the integer values from $-l$ to l , so that the degeneracy of the energy state is $2l+1$.

Finally we want to demonstrate that Pauli's treatment of the H-Atom leads directly to the method developed by Fock². We hereby take advantage of Bargmann's work³, which we adopt for two spatial dimensions.

Consider the following calculations in a Euclidean space of dimensionality D, in particular, D=3. Can we derive the result (35), using the 3-D angular momentum directly?

We found already that

$$L_{ab} = \xi_a \frac{1}{i} \frac{\partial}{\partial \xi_b} - \xi_b \frac{1}{i} \frac{\partial}{\partial \xi_a}.$$

Squaring this expression, we obtain

$$\frac{1}{2} \sum_{a,b} L_{ab}^2 = -\frac{1}{2} \sum_{a,b} (\xi_a \partial_b - \xi_b \partial_a)^2 = -\sum_{a,b} (\xi_a \partial_b \xi_a \partial_b - \xi_a \partial_b \xi_b \partial_a). \quad (36)$$

Let us rewrite this equation in terms of

$$\sum_a \xi_a^2 = \xi^2, \quad \sum_a \partial_a^2 = \partial^2, \quad \sum_a \xi_a \partial_a = \xi \cdot \partial.$$

Notice $\partial_b \xi_a = \xi_a \partial_b + \delta_{ba}$, $\partial_b \xi_b = \xi_b \partial_b + \delta_{bb}$. Then

$$\begin{aligned} \frac{1}{2} \sum_{a,b} L_{ab}^2 &= -\sum_{a,b} (\xi_a \xi_a \partial_b \partial_b + \delta_{ba} \xi_a \partial_b - \xi_b \underbrace{\xi_a \partial_b}_{=\partial_b \xi_a - \delta_{ab}} \partial_a - \delta_{bb} \xi_a \partial_a) \\ &= -\xi^2 \partial^2 + (\xi \cdot \partial)^2 + (D-2) \xi \cdot \partial. \end{aligned} \quad (37)$$

We now want to find eigenvalue solutions for this differential operator. Let f be a solution with $\partial^2 f = 0$ with f homogeneous in x to some degree: $(\xi \cdot \partial)f = d f$, $d = 0, 1, 2, \dots$. Then consider the special case $D=3$:

$$(\xi \cdot \partial) S(\xi) = l S(\xi), \quad \partial^2 S = 0.$$

This choice reduces Eq. (37) to

$$\frac{1}{2} \sum_{a,b} L_{ab}^2 S(\xi) = [l^2 + (3-2)l] S(\xi) = l(l+1) S(\xi),$$

where now $S(\xi)$ are the well-known spherical harmonics, and so indeed we come back to Eq. (35). Thus we have solved our eigenvalue problem,

$$\frac{1}{2} \sum_{a,b} L_{ab}^2 S(\xi) = \lambda S(\xi),$$

with

$$\lambda = l(l+1) (\hbar^2), \quad S(\xi) = Y_{lm}(\Omega).$$

3 The 2-D Green's Function of the H-Atom on Momentum Space

We begin the discussion of the 2-D hydrogen atom with the Green's function equation in momentum space:

$$\langle \mathbf{p} | \left(E - H_0 + \frac{Ze^2}{r} \right) G | \mathbf{p}' \rangle = \langle \mathbf{p} | \mathbf{p}' \rangle. \quad (38)$$

Here we recall Eq. (1). H_0 is the Hamiltonian for the free particle: $H_0 = \frac{\mathbf{p}^2}{2m}$.

Obviously we need

$$\langle \mathbf{p} | \left(\frac{1}{r} G \right) | \mathbf{p}' \rangle = \int d^2 p'' \langle \mathbf{p} | \left(\frac{1}{r} \right) | \mathbf{p}'' \rangle \langle \mathbf{p}'' | G | \mathbf{p}' \rangle. \quad (39)$$

One verifies directly¹³ that

$$\langle \mathbf{p} | \left(\frac{1}{r} \right) | \mathbf{p}'' \rangle = \frac{1}{2\pi |\mathbf{p} - \mathbf{p}''|}, \quad (40)$$

so that

$$\langle \mathbf{p} | \left(\frac{1}{r} G \right) | \mathbf{p}' \rangle = \frac{1}{2\pi} \int d^2 p'' \frac{1}{|\mathbf{p} - \mathbf{p}''|} G(\mathbf{p}'', \mathbf{p}'), \quad (41)$$

and from Eq. (38):

$$\left(E - \frac{\mathbf{p}^2}{2m} \right) G(\mathbf{p}, \mathbf{p}') + \frac{Z e^2}{2\pi} \int d^2 p'' \frac{1}{|\mathbf{p} - \mathbf{p}''|} G(\mathbf{p}'', \mathbf{p}') = \delta^2(\mathbf{p} - \mathbf{p}'). \quad (42)$$

This is our fundamental Green's function equation which we want to solve, assuming

$$E = -\frac{p_0^2}{2m}, \quad (43)$$

i.e., we restrict ourselves for the time being to $E < 0$ and define $p_0 = \sqrt{-2mE}$.

At this stage we introduce the Fock-sphere once again and set the 2-D momentum space into one-to-one correspondence to the surface of the unit sphere in 3-D:

$$\xi_1 \equiv \frac{x}{p_0} = \sin \theta \cos \phi = \frac{2p_0 p_x}{\lambda(p)}, \quad \lambda(p) = p_0^2 + p^2 \quad (44)$$

$$\xi_2 \equiv \frac{y}{p_0} = \sin \theta \sin \phi = \frac{2p_0 p_y}{\lambda(p)} \quad (45)$$

$$\xi_0 \equiv \frac{z}{p_0} = \cos \theta = \frac{p_0^2 - p^2}{\lambda(p)}, \quad (46)$$

where

$$\xi_0^2 + \xi^2 = 1 = \left(\frac{z}{p_0} \right)^2 + \left(\frac{x}{p_0} \right)^2 + \left(\frac{y}{p_0} \right)^2. \quad (47)$$

The area element on the unit sphere is

$$d\Omega = \sin \theta \, d\theta \, d\phi = -d(\cos \theta) \, d\phi,$$

and upon using Eq. (46),

$$\frac{d \cos \theta}{dp} = -\frac{(2p_0)^2 p}{\lambda^2},$$

we obtain

$$d\Omega = \left(\frac{2p_0}{\lambda} \right)^2 p \, dp \, d\phi = \left(\frac{2p_0}{\lambda} \right)^2 d^2 p. \quad (48)$$

One can also write

$$\begin{aligned}
d\Omega &= 2d^3\xi \delta(\xi^2 - 1) \\
&\equiv 2d^2\xi d\xi_0 \delta[\xi_0^2 - (1 - \boldsymbol{\xi}^2)] \\
&= 2d^2\xi \frac{d\xi_0^2}{2|\xi_0|} \delta[\xi_0^2 - (1 - \boldsymbol{\xi}^2)] = \frac{d^2\xi}{|\xi_0|}, \quad \xi_0 = \sqrt{1 - \boldsymbol{\xi}^2} = \text{Eq. (46)}.
\end{aligned}$$

It is easy to check that indeed $\int d\Omega = 4\pi$. The delta function connecting two points on the unit sphere is, according to Eq. (48),

$$\delta(\Omega - \Omega') = \left(\frac{\lambda}{2p_0}\right)^2 \delta(\mathbf{p} - \mathbf{p}'), \quad (49)$$

and the distance squared between two points ξ, ξ' on the Fock surface ($\xi \cdot \xi' = \cos \gamma$) is given by

$$\begin{aligned}
\left(2 \sin \frac{\gamma}{2}\right)^2 &= (\xi - \xi')^2 = (\xi_0 - \xi'_0)^2 + (\boldsymbol{\xi} - \boldsymbol{\xi}')^2 \\
&= \left(\frac{x - x'}{p_0}\right)^2 + \left(\frac{y - y'}{p_0}\right)^2 + \left(\frac{z - z'}{p_0}\right)^2 \stackrel{(44)-(46)}{=} \frac{4p_0^2}{\lambda(p)\lambda(p')} (\mathbf{p} - \mathbf{p}')^2. \quad (50)
\end{aligned}$$

Then, if we define

$$G(\mathbf{p}, \mathbf{p}') = -8mp_0^2 \frac{1}{\lambda(p)^{3/2}} \Gamma(\Omega, \Omega') \frac{1}{\lambda(p')^{3/2}}, \quad (51)$$

we can rewrite Eq. (42) in the form

$$\Gamma(\Omega, \Omega') - \frac{Ze^2 m}{2\pi p_0} \int d\Omega'' \frac{1}{|\xi - \xi''|} \Gamma(\Omega'', \Omega') = \delta(\Omega - \Omega'). \quad (52)$$

Upon using the Green's function equation

$$-\partial^2 D(\xi - \xi') = \delta(\xi - \xi'), \quad (53)$$

where

$$D(\xi - \xi') = \frac{1}{4\pi} \frac{1}{|\xi - \xi'|}, \quad (54)$$

the surface integral equation (52) becomes

$$\Gamma(\Omega, \Omega') - 2\nu \int d\Omega'' D(\xi - \xi'') \Gamma(\Omega'', \Omega') = \delta(\Omega - \Omega'), \quad (55)$$

with

$$\nu = \frac{Ze^2m}{p_0}. \quad (56)$$

Here it is useful to recall¹⁴

$$\frac{1}{|\xi - \xi'|} \Big|_{|\xi|=1=|\xi'|} \left(= \frac{1}{2 \sin \frac{\gamma}{2}} \right) = \sum_{l,m} \frac{2\pi}{l + \frac{1}{2}} Y_{lm}(\Omega) Y_{lm}^*(\Omega'). \quad (57)$$

Then the Green's function (54) is exhibited as

$$D(\xi - \xi') = \frac{1}{2} \sum_{l,m} \frac{1}{l + \frac{1}{2}} Y_{lm}(\Omega) Y_{lm}^*(\Omega'). \quad (58)$$

Also remember the completeness relation of the spherical harmonics:

$$\delta(\Omega - \Omega') = \sum_{l,m} Y_{lm}(\Omega) Y_{lm}^*(\Omega'), \quad (59)$$

and the normalization

$$\int d\Omega'' Y_{lm}^*(\Omega'') Y_{l'm'}(\Omega'') = \delta_{ll'} \delta_{mm'}. \quad (60)$$

With this information one can easily verify that (55) is solved by

$$\Gamma(\Omega, \Omega') = \sum_{l,m} \frac{Y_{lm}(\Omega) Y_{lm}^*(\Omega')}{1 - \frac{\nu}{l + \frac{1}{2}}}. \quad (61)$$

The poles of Eq. (61) yield the energy eigenvalues:

$$1 - \frac{\nu}{l + \frac{1}{2}} = 0, \quad \nu = \frac{Ze^2m}{p_0}$$

or $\nu = l + \frac{1}{2}$:

$$\nu^2 = \frac{m^2 Z^2 e^4}{p_0^2} = \frac{m^2 Z^2 e^4}{-2mE} = \left(l + \frac{1}{2} \right)^2.$$

So we obtain once again ($\hbar = 1$):

$$E_l = -\frac{mZ^2e^4}{2\left(l + \frac{1}{2}\right)^2}, \quad l = 0, 1, 2, \dots, \quad (62)$$

$$\text{or} \quad E_n = -\frac{mZ^2e^4}{2\left(n - \frac{1}{2}\right)^2}, \quad n = 1, 2, \dots. \quad (63)$$

The normalized wave function follows from our result Eq. (61):

$$\int d\Omega |Y|^2 = 1 = \int \left(\frac{2p_0}{\lambda(p)}\right)^2 d^2p |Y|^2. \quad (64)$$

Furthermore, $\int d\Omega \xi_0 |Y|^2 = 0$, since under $\xi_k \rightarrow -\xi_k$, $k = 0, 1, 2$, we have $Y \rightarrow (-1)^l Y$: $|Y|^2 \rightarrow |Y|^2$. Hence

$$0 = \int \left(\frac{2p_0}{\lambda(p)}\right)^2 d^2p \frac{p_0^2 - p^2}{\lambda(p)} |Y|^2. \quad (65)$$

Adding Eqs. (64) and (65) we obtain

$$1 = \int \left(\frac{2p_0}{\lambda(p)}\right)^2 d^2p \underbrace{\left[1 + \frac{p_0^2 - p^2}{\lambda}\right]}_{=\frac{2p_0^2}{\lambda}} |Y|^2 = \int d^2p 8p_0^4 \frac{|Y|^2}{\lambda(p)^3}.$$

This result can be used to write for the normalized momentum wave function:

$$\psi_{lm}(\mathbf{p}) = \frac{\sqrt{8}p_0^2}{(p_0^2 + p^2)^{3/2}} Y_{lm}(\Omega_p), \quad \int d^2p |\psi|^2 = 1, \quad (66)$$

where $p_0 = \frac{mZe^2}{l + \frac{1}{2}}$.

We now want to write $\Gamma(\Omega, \Omega')$ in a form that will be easy to continue analytically. To do this, we note the generating function for the Legendre polynomials

$$\frac{1}{\sqrt{1 - 2x\mu + \mu^2}} = \sum_{l=0}^{\infty} \mu^l P_l(x), \quad |\mu| < 1$$

and $P_l(\cos \gamma) = \frac{4\pi}{2l+1} \sum_m Y_{lm}(\Omega) Y_{lm}^*(\Omega'),$

so that

$$\frac{1}{\sqrt{1 - 2\mu \cos \gamma + \mu^2}} = \sum_{l=0}^{\infty} \mu^l P_l(\cos \gamma) = \sum_l \mu^l \frac{4\pi}{2l+1} \sum_m Y_{lm}(\Omega) Y_{lm}^*(\Omega').$$

Using $-2 \cos \gamma = -2\xi \cdot \xi' = (\xi - \xi')^2 - 2$, $|\xi| = |\xi'| = 1$, we get:

$$1 - 2\mu \cos \gamma + \mu^2 = 1 + \mu[(\xi - \xi')^2 - 2] + \mu^2 = (1 - \mu^2) + \mu(\xi - \xi')^2.$$

This allows us to write:

$$\frac{1}{\sqrt{(1 - \mu^2) + \mu(\xi - \xi')^2}} = \sum_{l,m} \mu^l \frac{4\pi}{2l+1} Y_{lm}(\Omega) Y_{lm}^*(\Omega'). \quad (67)$$

Note, incidentally, that for $\xi = \xi'$ we obtain

$$\begin{aligned} \frac{1}{1 - \mu} \underbrace{\int d\Omega}_{=4\pi} &= 4\pi \sum_{l,m} \frac{\mu^l}{2l+1} \int d\Omega |Y_{lm}(\Omega)|^2, \\ \text{or} \quad \frac{1}{1 - \mu} &= \sum_l \frac{\mu^l}{2l+1} \int d\Omega \sum_m |Y_{lm}(\Omega)|^2 = \sum_{l=0}^{\infty} \frac{\mu^l}{2l+1} m(l), \end{aligned}$$

which again yields the multiplicity of the quantum number l :

$$\begin{aligned} m(l) &= 2l+1, & l &= 0, 1, 2, \dots \\ \text{or} \quad m(n) &= 2n-1, & n &= 1, 2, \dots \end{aligned}$$

Now we return to our main result Eq. (61). Use of the identity

$$\left(1 - \frac{\nu}{l + \frac{1}{2}}\right)^{-1} = 1 + \frac{\nu}{l + \frac{1}{2}} + \nu^2 \frac{1}{(l + \frac{1}{2})(l + \frac{1}{2} - \nu)}$$

and the integral representation (valid for $\nu < 1/2$):

$$\frac{1}{(l+1) - (\nu + 1/2)} = \int_0^1 d\mu \mu^{-(\nu+1/2)} \mu^l$$

produces

$$\Gamma(\Omega, \Omega') = \delta(\Omega - \Omega') + \frac{\nu}{2\pi} \frac{1}{|\xi - \xi'|} + \frac{\nu^2}{2\pi} \int_0^1 d\mu \mu^{-(\nu+1/2)} \frac{1}{\sqrt{(1 - \mu^2) + \mu(\xi - \xi')^2}}. \quad (68)$$

Performing an integration by parts yields still another representation for Γ :

$$\Gamma(\Omega, \Omega') = \delta(\Omega - \Omega') + \frac{\nu}{2\pi} \int_0^1 d\mu \mu^{-\nu} \frac{d}{d\mu} \frac{\mu^{\frac{1}{2}}}{\sqrt{(1-\mu^2) + \mu(\xi - \xi')^2}}. \quad (69)$$

Let us pause for a moment and look at the pole structure of Eq. (68):

$$\begin{aligned} \int_0^1 d\mu \mu^{-(\nu+1/2)} \frac{1}{\sqrt{(1-\mu^2) + \mu(\xi - \xi')^2}} &= \left. \frac{1}{\frac{1}{2} - \nu} \frac{1}{\sqrt{(1-\mu^2) + \mu(\xi - \xi')^2}} \right|_{\mu=0} \\ &\quad - \int_0^1 d\mu \frac{\mu^{1/2-\nu} - 1}{\frac{1}{2} - \nu} \frac{d}{d\mu} \frac{1}{\sqrt{(1-\mu^2) + \mu(\xi - \xi')^2}}. \end{aligned} \quad (70)$$

Introducing Eq. (68) in Eq. (51) we obtain

$$\begin{aligned} G(\mathbf{p}, \mathbf{p}') &= -8mp_0^2 \frac{1}{\lambda(p)^{3/2}} \left[\delta(\Omega - \Omega') + \frac{\nu}{2\pi} \frac{1}{|\xi - \xi'|} \right. \\ &\quad \left. + \frac{\nu^2}{2\pi} \int_0^1 d\mu \mu^{-(\nu+1/2)} \frac{1}{\sqrt{(1-\mu^2) + \mu(\xi - \xi')^2}} \right] \frac{1}{\lambda(p')^{3/2}}. \end{aligned} \quad (71)$$

Here, we consider only the μ -integral term which yields, with the aid of Eq. (70):

$$G(\mathbf{p}, \mathbf{p}') = \dots - 8mp_0^2 \frac{1}{\lambda(p)^{3/2}} \frac{\nu^2}{2\pi} \left[\frac{1}{\frac{1}{2} - \nu} - \int_0^1 d\mu \frac{\mu^{1/2-\nu} - 1}{\frac{1}{2} - \nu} \frac{d}{d\mu} \frac{1}{\sqrt{(1-\mu^2) + \mu(\xi - \xi')^2}} \right] \frac{1}{\lambda(p')^{3/2}}. \quad (72)$$

The pole contribution is obviously contained in

$$G(\mathbf{p}, \mathbf{p}') = -8mp_0^2 \frac{1}{\lambda(p)^{3/2}} \frac{\nu^2}{2\pi} \frac{1}{\frac{1}{2} - \nu} \frac{1}{\lambda(p')^{3/2}} + \dots,$$

where $\nu = \frac{1}{2}$ corresponds to the ground state $n = 1$: $E_1 = -2mZ^2e^4$.

Recall $\frac{1}{2} = \nu = \frac{Ze^2m}{p_0}$: $p_0 = 2Ze^2m$, $\nu^2 = \frac{Z^2e^4m}{-2E}$, so that $\frac{1}{\frac{1}{2}-\nu} = \frac{\frac{1}{2}+\nu}{\frac{1}{4}-\nu^2}$,

$$\begin{aligned} \Rightarrow \quad \frac{1}{\frac{1}{4}-\nu^2} &= \frac{4}{1-4\nu^2} = \frac{4}{1+\frac{2Z^2e^4m}{E}} = \frac{4E}{E+2me^4Z^2} = \frac{4E}{E-E_1} \\ &= -\frac{2mZ^2e^4}{\nu^2} \frac{1}{E-E_1}, \\ \text{and } \frac{\nu^2}{2\pi} \frac{1}{\frac{1}{2}-\nu} &= \frac{1}{2\pi} (-2mZ^2e^4) \frac{1}{E-E_1}. \end{aligned} \quad (73)$$

We need this result in Eq. (72):

$$-8mp_0^2 \frac{\nu^2}{2\pi} \frac{1}{\frac{1}{2}-\nu} = \frac{2p_0^4}{\pi} \frac{1}{E-E_1}.$$

Hence we obtain:

$$G_E(\mathbf{p}, \mathbf{p}') = \frac{1}{\lambda(p)^{3/2}} \frac{2p_0^4}{\pi} \frac{1}{E-E_1} \frac{1}{\lambda(p')^{3/2}} + \dots \quad (74)$$

The remaining integral is defined for all ν such that $\text{Re } \nu < \frac{3}{2}$. This process can be repeated as often as necessary to isolate more poles and extend the acceptable region for ν .

So far we have been interested in bound states. But from now on we will be interested in scattering states. Hence we extend ν analytically to complex values, in particular to the imaginary axis. So let us define

$$\eta = -i\nu = \frac{mZe^2}{k}, \quad k = \sqrt{2mE} \quad (E > 0). \quad (75)$$

Again we go back to our fundamental equation (51) and use Eq. (69) for $\Gamma(\Omega, \Omega')$:

$$\begin{aligned} G(\mathbf{p}, \mathbf{p}') &= -8mp_0^2 \frac{1}{\lambda(p)^{\frac{3}{2}}} \left[\delta(\Omega - \Omega') + \frac{\nu}{2\pi} \int_0^1 d\mu \mu^{-\nu} \frac{d}{d\mu} \frac{\mu^{\frac{1}{2}}}{\sqrt{(1-\mu^2) + \mu(\xi - \xi')^2}} \right] \frac{1}{\lambda(p')^{\frac{3}{2}}} \\ &= -8mp_0^2 \frac{1}{\lambda(p)^{3/2}} \left(\frac{\lambda(p)}{2p_0} \right)^2 \delta(\mathbf{p} - \mathbf{p}') \frac{1}{\lambda(p')^{3/2}} + \dots \\ &= -2m \frac{1}{\lambda(p)} \delta(\mathbf{p} - \mathbf{p}') + \dots \end{aligned} \quad (76)$$

Here we write $\lambda(p) = p_0^2 + p^2 = p^2 - 2mE = 2m \left(\frac{\mathbf{p}^2}{2m} - E \right) = 2m(T - E)$, so that

$G(\mathbf{p}, \mathbf{p}') = \frac{1}{E-T} \delta(\mathbf{p} - \mathbf{p}') + \dots$, with $T = \frac{\mathbf{p}^2}{2m}$. This amounts to writing ($\nu = i\eta$, $p_0 = -ik$)

$$\begin{aligned} G(\mathbf{p}, \mathbf{p}') &= \frac{\delta(\mathbf{p} - \mathbf{p}')}{E - T} - 8mp_0^2 \frac{1}{\lambda(p)^{3/2}} \frac{mZe^2}{2\pi p_0} \int_0^1 d\mu \mu^{-i\eta} \frac{d}{d\mu} \frac{\mu^{\frac{1}{2}}}{\sqrt{(1-\mu^2) + \mu(\xi - \xi')^2}} \frac{1}{\lambda(p')^{3/2}} \\ &= \frac{\delta(\mathbf{p} - \mathbf{p}')}{E - T} \\ &\quad - \frac{Ze^2}{\pi} p_0 \frac{1}{E - T} \int_0^1 d\mu \mu^{-i\eta} \frac{d}{d\mu} \frac{\mu^{\frac{1}{2}}}{\sqrt{[(1-\mu^2) + \mu(\xi - \xi')^2] \lambda(p) \lambda(p')}} \frac{1}{E - T'}. \end{aligned} \quad (77)$$

using relation (50)

$$(\xi - \xi')^2 = \frac{4p_0^2}{\lambda(p)\lambda(p')} (\mathbf{p} - \mathbf{p}')^2,$$

the square root in (77) can also be rewritten as

$$2p_0 \sqrt{(\mathbf{p} - \mathbf{p}')^2 \mu - \frac{m}{2E} (E - T)(E - T')(1 - \mu)^2}, \quad (78)$$

so that we now have

$$\begin{aligned} G(\mathbf{p}, \mathbf{p}') &= \frac{\delta(\mathbf{p} - \mathbf{p}')}{E - T} \\ &\quad - \frac{Ze^2}{2\pi} p_0 \frac{1}{E - T} \int_0^1 d\mu \mu^{-i\eta} \frac{d}{d\mu} \frac{\mu^{\frac{1}{2}}}{\sqrt{(\mathbf{p} - \mathbf{p}')^2 \mu - \frac{m}{2E} (E - T)(E - T')(1 - \mu)^2}} \frac{1}{E - T'}. \end{aligned} \quad (79)$$

Performing an integration by parts, it is easy to show that the μ -integral can be written as

$$\frac{1}{2p_0 |\mathbf{p} - \mathbf{p}'|} + i\eta \int_0^1 d\mu \mu^{-(i\eta+1/2)} \frac{1}{2p_0 \sqrt{(\mathbf{p} - \mathbf{p}')^2 \mu - \frac{m}{2E} (E - T)(E - T')(1 - \mu)^2}}. \quad (80)$$

Because the scattering is characterized by

$$(E - T) \sim 0 \sim (E - T'), \quad (\mathbf{p} - \mathbf{p}')^2 > 0, \quad (81)$$

we can replace the square root in (80) (read together with the μ -integral) by

$$\frac{1}{\sqrt{(\mathbf{p} - \mathbf{p}')^2 \mu - \frac{m}{2E} (E - T)(E - T')}} = \frac{1}{i \sqrt{\frac{m}{2E} (E - T)(E - T')}} \frac{1}{\sqrt{1 - \beta\mu}} \quad (82)$$

with

$$\beta := \frac{(\mathbf{p} - \mathbf{p}')^2}{\frac{m}{2E}(E - T)(E - T')}. \quad (83)$$

So the μ -integral in Eq. (79) is given by

$$\frac{1}{2p_0|\mathbf{p} - \mathbf{p}'|} + \frac{\eta}{2p_0} \frac{1}{\sqrt{\frac{m}{2E}(E - T)(E - T')}} \int_0^1 d\mu \mu^{-i\eta - \frac{1}{2}} (1 - \beta\mu)^{-\frac{1}{2}}. \quad (84)$$

In the limit of large β the integral in (84) may be computed with some formulas given in Ref. 15. As an intermediate result for our Green's function we then obtain

$$G(\mathbf{p}, \mathbf{p}') = \frac{\delta(\mathbf{p} - \mathbf{p}')}{E - T} - G_0^C(\mathbf{p}) \left[\frac{m}{(2\pi)^2 \sqrt{\pi}} \frac{e^{i\eta \ln \frac{(\mathbf{p} - \mathbf{p}')^2}{4p_0^2}}}{|\mathbf{p} - \mathbf{p}'|} \frac{\Gamma(1/2 - i\eta)}{\Gamma(i\eta)} i^{-2i\eta} \right] G_0^C(\mathbf{p}') \quad (85)$$

where

$$G_0^C(\mathbf{p}) = \frac{\sqrt{2\pi i k}}{m} \frac{i^{(1+i\eta)} \Gamma(1 + i\eta)}{E - T} e^{-i\eta \ln \frac{E - T}{4E}}. \quad (86)$$

Incidentally, when we take the Fourier transform of this expression we get (for large r):

$$\int d^2p e^{i\mathbf{p} \cdot \mathbf{r}} G_0^C(\mathbf{p}) = \frac{1}{\sqrt{r}} e^{i(kr + \eta \ln 2kr)}. \quad (87)$$

Since $(\mathbf{p} - \mathbf{p}')^2 = 4p_0^2 \sin^2 \frac{\phi}{2}$, where we assumed $p^2 = p'^2 \simeq p_0^2 = -2mE = -k^2$, $p_0 = -ik$, we have $|\mathbf{p} - \mathbf{p}'| = \sqrt{4p_0^2 \sin^2 \frac{\phi}{2}}$, so that the square brackets in Eq. (85) take the value

$$\frac{m}{2\pi \sqrt{i k}} \frac{1}{(2\pi)^{3/2}} \tilde{f}(\phi) i^{-2i\eta}, \quad (88)$$

with

$$\tilde{f}(\phi) = \frac{e^{i\eta \ln \sin^2 \frac{\phi}{2}}}{\sqrt{2ik \sin^2 \frac{\phi}{2}}} \frac{\Gamma(1/2 - i\eta)}{\Gamma(i\eta)}. \quad (89)$$

If we then choose the phase such that

$$e^{i \left[\arg \Gamma(1/2 - i\eta) - \arg \Gamma(i\eta) \right]} = (ik)^{\frac{1}{2}} i^{2i\eta},$$

we finally obtain for the 2-D Coulomb Green's function:

$$G_E(\mathbf{p} - \mathbf{p}') = \frac{\delta(\mathbf{p} - \mathbf{p}')}{E - T} - G_0^C(\mathbf{p}) \frac{m}{2\pi} \frac{1}{(2\pi)^{\frac{3}{2}}} f(\phi) G_0^C(\mathbf{p}'), \quad (90)$$

where

$$f(\phi) = \frac{|\Gamma(1/2 - i\eta)|}{|\Gamma(i\eta)|} \frac{e^{i\eta \ln \sin^2 \frac{\phi}{2}}}{\sqrt{2ik \sin^2 \frac{\phi}{2}}}. \quad (91)$$

This is the scattering amplitude that is needed to compute the differential cross section

$$\sigma(\phi) = |f(\phi)|^2. \quad (92)$$

Using the formulas

$$|\Gamma(i\eta)|^2 = \frac{\pi}{\eta \sinh \eta\pi}, \quad |\Gamma(1/2 - i\eta)|^2 = \frac{\pi}{\cosh \eta\pi},$$

we get

$$\sigma(\phi) = \frac{\eta \tanh \eta\pi}{2k \sin^2 \frac{\phi}{2}}, \quad (93)$$

and with $\eta = \frac{mZe^2}{k}$ we arrive at

$$\sigma(\phi) = \frac{mZe^2}{2k^2 \sin^2 \frac{\phi}{2}} \tanh \frac{\pi mZe^2}{k}. \quad (94)$$

In the high energy limit ($k \rightarrow \infty$, $\eta \rightarrow 0$) we then obtain

$$\begin{aligned} \sigma(\phi) &= \frac{mZe^2}{2k^2 \sin^2 \frac{\phi}{2}} \frac{\pi mZe^2}{k} & k &= \frac{mv}{\hbar} \\ &= \frac{\pi(Ze^2)^2}{2\hbar m v^3 \sin^2 \frac{\phi}{2}}. \end{aligned} \quad (95)$$

This agrees with the Born approximation. Our result was also found in Ref. 16.

4 Conclusion

In this paper we have studied the quantum mechanical Coulomb problem in two spatial dimensions. Although it is true that the three-dimensional analogue is the more important – since physical – one, it seems to us that the two-dimensional model helps immensely to understand the mathematical aspect of the real three-dimensional case. Spherical harmonics on the two-sphere are certainly more familiar than the ones on the three-sphere. Following the strategy initiated by Fock, Pauli and Schwinger, we were able to solve the two-dimensional Coulomb problem analytically, i.e., we presented the exact Green's function for the two-dimensional hydrogen atom. Exact formulas were then given for both the discrete and the continuous parts of the spectrum. We hope that by studying the present paper the reader will not have any problem reproducing Schwinger's superb paper¹ on the same subject.

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References

- [1] J. Schwinger, *Coulomb Green's function*, J. Math. Phys. **5**, 1606-1608 (1964).
- [2] V.A.Fock, *Zur Theorie des Wasserstoffatoms*, Zeit. Phys. **98**, 145-154 (1935).
- [3] V. Bargmann, Zeit. Phys. **99**, 576-582 (1936).
- [4] B. Podolsky and L. Pauling, *The momentum distribution in hydrogen atoms*, Phys. Rev. **34**, 109-116 (1929).
- [5] J.J. Klein, *Eigenfunctions of the Hydrogen Atom in Momentum Space*, Am. J. Phys. **34**, 1039-1042 (1966).
- [6] O.L. de Lange and R.E. Raab, *An operator solution for the hydrogen atom with application to the momentum representation*, Am. J. Phys. **55**, 913-917 (1987).
- [7] B.R. Holstein, *Quantum Mechanics in Momentum Space: The Coulomb System*, Am. J. Phys. **63**, 710-716 (1995).
- [8] G.S. Adkins, *Derivation of the Schrödinger-Coulomb Green's function from the scattering expansion*, Nuov. Cim. B **97**, 99-107 (1987).

- [9] M. Lieber, *The Coulomb Green's function*, in *Relativity, Quantum Electrodynamics and Weak Interaction Effects in Atoms*, edited by W. Johnson, P. Mohr, and J. Sucher, AIP Conference Proceedings No. 189 (AIP, New York, 1989), 445-459 (1989).
- [10] M. Lieber, *$O(4)$ symmetry of the hydrogen atom and the Lamb shift*, Phys. Rev. **174**, 2037-2054 (1968).
- [11] W. Pauli, Jr., *Über das Wasserstoffspektrum vom Standpunkt der Neuen Quantenmechanik*, Zeit. Phys. **36**, 336-363 (1926).
- [12] X.L. Yang, M. Lieber, and F.T. Chan, *The Runge-Lenz vector for the two-dimensional hydrogen atom*, Am. J. Phys. **59**, 231-232 (1991).
- [13] T.-I. Shibuya and C.E. Wulfman, *The Kepler problem in two-dimensional momentum space*, Am. J. Phys. **33**, 570-574 (1965).
- [14] J.D. Jackson, *Classical Electrodynamics*, Second Edition, (John Wiley&Sons, Inc., New York, 1975), p.102, (3.70) or
J. Schwinger, L.L. DeRaad, Jr., K.A. Milton, W.-y. Tsai, *Classical Electrodynamics*, (Perseus Books, Reading, Massachusetts, 1998), p.246, (21.26).
- [15] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products*, (Academic Press, New York, 1965), Refs. 3.197.3, 9.132.2 (1965).
- [16] G. Barton, *Rutherford scattering in two dimensions*, Am. J. Phys. **51**, 420-422 (1983);
Q.-g. Lin, *Scattering by a Coulomb field in two dimensions*, Am. J. Phys. **65**, 1007-1009 (1997).